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Computation Of Some Zamolodchikov Volumes, With An Application

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Abstract: We compute the Zamolodchikov volumes of some moduli spaces of conformal field theories with target spaces K3, T4, and their symmetric products. As an application we argue that sequences of conformal field theories, built from products of such symmetric products, almost never have a holographic dual with weakly coupled gravity. August 6, 2015

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1. Introduction And Motivation

Physicists often speak of the set of all two-dimensional conformal field theories as a "moduli space of conformal field theories." It is believed that one can make sense of this set as a topological space and that, moreover, the generic point in this space has a smooth neighborhood locally modeled on a manifold. It was noted by Zamolodchikov that at such smooth points the moduli space has a *canonical metric* [30]. In many cases of interest in string theory these moduli spaces have finite volume in the Zamolodchikov metric, a fact of some importance when one applies statistical ideas to string compactification [1, 10, 11, 12, 18]. In a recent paper [5] the finiteness of the Zamolodchikov volume for certain families of 2d CFT's has again played an important role. The present note should be regarded as an addendum to [5].

In this note we compute the Zamolodchikov volumes for the 2d nonlinear sigma model whose target space is (the hyperkähler resolution of) a symmetric product of K3 surfaces or four-dimensional tori. Moreover, we return to the statistical considerations of [5] and show in a rather precise sense that the set of sequences of conformal field theories with weakly coupled gravitational holographic duals, when drawn from the ensembles described in Section §3 below, is of measure zero.

2. Zamolodchikov Metric

If a two-dimensional conformal field theory $\mathcal{C}$ is a smooth point in a moduli space $\mathcal{M}$ of CFT's then there is a canonical isomorphism between the tangent space $T_{\mathcal{C}}\mathcal{M}$ and the vector space $V^{1,1}$ of exactly marginal $(1,1)$ operators in $\mathcal{C}$. For small ϵ the isomorphism takes $\epsilon\mathcal{O} \in V^{1,1}$ to the tangent vector defined by deforming correlation functions to:

$$\left\langle \prod_i \Phi_i \right\rangle \to \left\langle e^{-\int \epsilon\mathcal{O}} \prod_i \Phi_i \right\rangle \tag{2.1}$$

In the case where the CFT's are defined by an action we can be a bit more precise. We define the isomorphism

$$\Psi : V^{1,1} \to T_{\mathcal{C}}\mathcal{M} \tag{2.2}$$

as follows: If $S[t]$ is a one-parameter family of actions of conformal field theories (hence a path in $\mathcal{M}$) and $\frac{d}{dt}|_0 S[t] = \int \mathcal{O}$ then

$$\Psi(\mathcal{O}) = \frac{\partial}{\partial t}\bigg|_0 . \tag{2.3}$$

In these terms, the Zamolodchikov metric is then defined by saying that if $v = \Psi(\mathcal{O})$ then

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\rangle := g_{\rm Z}(v,v)\frac{d^2z_1 d^2z_2}{|z_1 - z_2|^4} \tag{2.4}$$

where the LHS is the correlation function on the complex plane $\mathbb{C}$ with the unique $SL(2,\mathbb{C})$ invariant vacuum at $z = 0, \infty$, and $d^2z := dxdy = \frac{\mathrm{i}}{2} dz \wedge d\bar{z}$ with $z = x + \mathrm{i}y$. We will also denote the Zamolodchikov metric by ds^2_Z. Note that the Zamolodchikov metric has a <u>canonical</u> normalization. When the moduli space has a finite volume it is therefore meaningful to ask what that volume is. [1]

As an example, consider the Gaussian model for a periodic real scalar. The action for this model is

$$S = K \int d\phi * d\phi \tag{2.5}$$

with ϕ dimensionless and periodic (the period can be anything, so long as it is fixed) and K is a positive constant. One standard normalization in physics is to take $\phi \sim \phi + 2\pi$ and $K = \frac{R^2}{4\pi\alpha'}$. One readily computes that for the Gaussian model

$$ds^2_{\rm Z} = \frac{1}{(2\pi)^2}\left(\frac{dK}{K}\right)^2 = \frac{1}{\pi^2}\left(\frac{dR}{R}\right)^2 \tag{2.6}$$

The factor $\frac{1}{(2\pi)^2}$ comes from the basic fact that the Green's function in two dimensions is $\frac{1}{4\pi}\log|z_1 - z_2|^2$ so $\langle\phi(z_1)\phi(z_2)\rangle = \frac{1}{2\pi K}\log|z_1 - z_2|^2$. The moduli space is a half-line, and has an infinite volume.

[1] Of course, in some physics problems, a different normalization of the metric might be called for. The situation is similar to that of defining an Ad-invariant metric on a simple Lie algebra. The Ad-invariant metrics are unique up to scale. There is, however, a canonical normalization - given by the trace in the adjoint representation. In a metric $\lambda^2 ds^2_Z$ the volumes quoted in this paper are rescaled by a factor of $\lambda^{\mathcal{D}}$, where $\mathcal{D}$ is the real dimension.

Despite this unpromising beginning, the moduli spaces of chiral bosons in general do have finite volume moduli spaces. [2] We consider a theory of r right-moving and $r+8s$ left-moving chiral bosons based on even unimodular lattices $L_{r,s}$ of signature $(+1^{r+8s}, -1^{r})$ with $r > 0$ and $s \geq 0$. For concreteness, choose the quadratic form:

$$Q_{r,s} := U^{\oplus r} \oplus Q_8^{\oplus s} \tag{2.7}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.8}$$

and Q_8 is the Gram matrix of the E_8 lattice in some basis of simple roots. These theories come in moduli spaces generally called "Narain moduli spaces" in the string theory literature. Mathematically, they are the moduli spaces of embeddings of the lattices $L_{r,s}$ into the pseudo-Euclidean space $\mathbb{R}^{r+8s,r}$. These moduli spaces can be expressed as the double-coset

$$\mathcal{N}_{r+8s,r} := O_{\mathbb{Z}}(Q_{r,s}) \backslash O_{\mathbb{R}}(Q_{r,s}) / O(r+8s) \times O(r) \tag{2.9}$$

with

$$\begin{aligned} O_{\mathbb{Z}}(Q_{r,s}) &:= \{g \in GL(D, \mathbb{Z}) | g^{tr} Q_{r,s} g = Q_{r,s}\} \\ O_{\mathbb{R}}(Q_{r,s}) &:= \{g \in GL(D, \mathbb{R}) | g^{tr} Q_{r,s} g = Q_{r,s}\} \cong O(r+8s, r) \end{aligned} \tag{2.10}$$

where $D := 2r + 8s := 2d$. Note that they have dimension

$$\mathcal{D} = \dim \mathcal{N}_{r+8s,r} = r(r+8s). \tag{2.11}$$

For Narain moduli spaces the Zamolodchikov metric is a homogeneous metric induced by analytic continuation of a left-right invariant metric on the Lie algebra $so(D)$. This follows since the conformal field theories form an equivariant bundle of vertex operator algebras over $\mathcal{N}_{r+8s,r}$ (See, for examples, [22, 25] for the case $s = 0$.) The precise normalization of this homogeneous metric is of importance to us, and we determine it as follows:

We first recall how the Narain moduli space is parametrized in terms of the physical data of a conformal field theory with toroidal target space of dimension r: The physical data consists of a flat metric, a flat B-field, and a flat gauge connection for an E_8^s gauge group. We now relate these data to the embedding of $L_{r,s}$ into $\mathbb{R}^{r+8s;r}$. (At this point we no longer distinguish between $L_{r,s}$ and its embedded version.) The projection of a vector $p \in L_{r,s}$ to the components in the definite subspaces is denoted $p = (p_L; p_R)$. Therefore, the vectors can be written as:

$$\begin{pmatrix} p_L \\ p_R \end{pmatrix} = \mathcal{E} \vec{n} \tag{2.12}$$

where $\mathcal{E}$ is a $D \times D$ matrix and $\vec{n}$ is a D-component integral column vector representing the vector p in an integral basis of $L_{r,s}$. We have

$$\mathcal{E}^{tr} Q_0 \mathcal{E} = Q \tag{2.13}$$

[2] Actually, we should use the term "moduli stacks," but neither the reader nor the author will need to understand this term to make sense of this paper.

where

$$Q_0 = \begin{pmatrix} 1_{r+8s} & 0 \\ 0 & -1_r \end{pmatrix} \qquad\qquad Q = \begin{pmatrix} Q_{8s} & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix} \tag{2.14}$$

and Q_{8s} is the Gram matrix, in some basis, of some positive definite even unimodular lattice Γ_{8s} of dimension $8s$ (it does not matter which basis and which lattice). It is a positive definite symmetric even integral matrix of determinant one. Moreover, there exists $S = S^{tr} = S^{-1}$ so that

$$Q = \mathcal{S}^{tr} Q_0 \mathcal{S} \tag{2.15}$$

where

$$\mathcal{S} = \begin{pmatrix} f & 0 \\ 0 & S \end{pmatrix} \tag{2.16}$$

and f is a generating matrix for Q_{8s}, that is, a matrix formed from basis vectors for an embedding of Γ_{8s} into the Euclidean space $\mathbb{R}^{8s}$ with Gram matrix Q_{8s}:

$$f^{tr} f = Q_{8s}. \tag{2.17}$$

Note that $\mathcal{E}\mathcal{S}^{-1} \in O_{\mathbb{R}}(Q_0)$ and $\mathcal{S}^{-1}\mathcal{E} \in O_{\mathbb{R}}(Q)$ so $\mathcal{M}$ is diffeomorphic to an orthogonal group.

Denote the space of solutions to (2.13) by $\mathcal{M}$. A nice parametrization of the homogeneous space $O(r+8s) \times O(r) \backslash \mathcal{M}$ is given by a set of representatives $\mathcal{E}$ derived from the zeromodes for the left- and right-moving chiral fields using formulae from [15, 23]. The result is the following: Let e_1, e_2 be two invertible $r \times r$ matrices so that $e_1^{tr} e_1 = e_2^{tr} e_2$ is a positive definite symmetric matrix. Call it G^{-1}. Now let B be an arbitrary $r \times r$ antisymmetric matrix and form $E := G + B$. Let a be an arbitrary $8s \times r$ matrix. All the matrices e_1, e_2, B, a, f are over the real numbers. Now consider

$$\mathcal{E} = \begin{pmatrix} f & 0 & a \\ -\frac{1}{2} e_1 a^{tr} f & \frac{1}{2} e_1 & e_1 (E - \frac{1}{4} a^{tr} a) \\ -\frac{1}{2} e_2 a^{tr} f & \frac{1}{2} e_2 & -e_2 (E^{tr} + \frac{1}{4} a^{tr} a) \end{pmatrix} \tag{2.18}$$

The reader can readily check that this solves (2.13). The form of this expression is preserved by left-multiplication by $O(8s) \times O(r) \times O(r)$ and the map from $\mathcal{E}$ to $[\mathcal{E}]$ is a surjection onto $(O(r+8s) \times O(r)\mathcal{M}$.

Now we consider the subspace defined by $a = 0$, $B = 0$ and $e_1 = e_2 = \mathrm{Diag}\{R_1^{-1}, \ldots, R_r^{-1}\}$ where $R_i > 0$ are the radii of the square torus. The pullback of the Zamolodchikov metric to this subspace should be the Zamolodchikov metric for a product of r Gaussian models with radii R_i. Then we simply have

$$\mathrm{Tr}_D \mathcal{E}^{-1} d\mathcal{E} \otimes \mathcal{E}^{-1} d\mathcal{E} = 2 \sum_{i=1}^{d} \left(\frac{dR_i}{R_i} \right)^2 \tag{2.19}$$

Therefore, on all of $\mathcal{N}_{r+8s,r}$ the Zamolodchikov metric ds^2_{Z} corresponds to the homogeneous metric induced from

$$\frac{1}{2\pi^2} \mathrm{Tr}_D \mathcal{E}^{-1} d\mathcal{E} \otimes \mathcal{E}^{-1} d\mathcal{E}. \tag{2.20}$$

3. Some Ensembles Of $(4,4)$-Superconformal Field Theories

Consider a subspace of the set of conformal field theories, measurable in the Zamolodchikov volume form. When the total Zamolodchikov volume of this subspace is finite we can use the Zamolodchikov metric to define a probability measure on this set of CFT's.

In this paper we will apply this simple remark to ensembles of unitary $(4,4)$ superconformal field theories. We begin with a collection $\mathfrak{C}$ of CFT's written as a disjoint union over ensembles of definite central charge

$$\mathfrak{C} = \amalg_M \mathfrak{C}_M. \tag{3.1}$$

For each M, the ensemble at fixed central charge will be a disjoint union of connected components:

$$\mathfrak{C}_M = \amalg_\alpha \mathfrak{C}_{M,\alpha}. \tag{3.2}$$

We will be considering ensembles so that the total volume:

$$\mathrm{vol}\,(\mathfrak{C}_M) = \sum_\alpha \mathrm{vol}\,(\mathfrak{C}_{M,\alpha}) \tag{3.3}$$

is finite for fixed M. (The sum over M of the volumes $\mathrm{vol}\,(\mathfrak{C}_M)$ will not be finite, but we will nevertheless refer to $\mathfrak{C}$ as an "ensemble".)

In order to apply the techniques of Section §5 we will also want the volumes to be multiplicative:

$$\mathrm{vol}\,(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{vol}\,(\mathcal{C}_1)\mathrm{vol}\,(\mathcal{C}_2). \tag{3.4}$$

In general it is not true that the moduli space of a product of CFT's is the product of the moduli spaces. That is, in general $\mathcal{M}(\mathcal{C}_1 \times \mathcal{C}_2) \neq \mathcal{M}(\mathcal{C}_1) \times \mathcal{M}(\mathcal{C}_2)$. Toroidal models give simple examples of this inequality. Following [5] we define an ensemble of CFT's such that (3.4) does hold to be a *multiplicative ensemble*.

We will consider three distinct ensembles defined by the collection of two-dimensional superconformal field theories with target spaces of the form:

$$\left(S^1 X\right)^{n_1} \times \left(S^2 X\right)^{n_2} \times \cdots (S^r X)^{n_r}. \tag{3.5}$$

In the first ensemble we take X to be a $K3$ surface, in the second X is a four-dimensional torus $T4$, and in the third X can be either $K3$ or $T4$. The notation $S^m K3$ means the hyperkähler resolution of the symmetric product $\mathrm{Sym}^m(K3) = (K3)^m/S_m$. Choosing a complex structure compatible with the hyperkähler structure of $K3$, it is the Hilbert scheme of points $\mathrm{Hilb}^m(K3)$ endowed with a hyperkähler metric. Similarly we would like to consider the symmetric product of $T4$, but the resulting space has a nontrivial Betti number b_1 and would not define a multiplicative ensemble. Thus, to define $S^m T4$ we begin with the hyperkähler resolution $\pi : \mathrm{Hilb}^{m+1}(T4) \to \mathrm{Sym}^{m+1}(T4)$ and note that there is a map $f : \mathrm{Sym}^{m+1}(T4) \to T4$ given by taking the sum of the points. Then $S^m T4$ is defined to be the fiber above zero of $f \circ \pi$. It is a smooth compact simply connected hyperkähler manifold [2]. (Warning: $S^1 K3 = K3$ but $S^1 T4$ does not equal $T4$. Rather $S^1 T4$ is the Kummer surface derived from $T4$.) In all three cases the ensembles are multiplicative. The

reason for this, as noted in [5], is that for two Calabi-Yau manifolds X_1, X_2 with $h^{1,0} = 0$ the number of Kähler moduli $h^{1,1}$ and complex structure moduli $h^{n-1,1}$ is additive for the product $X_1 \times X_2$. [3]

For S^1K3 the moduli space is just the 80-dimensional space $\mathcal{N}_{20,4}$. For S^MK3 with $M > 1$ and S^MT4 with $M \geq 1$ the moduli space can be derived using the attractor mechanism, as pointed out in [9, 26]. We consider the subgroups of $O_{\mathbb{R}}(Q_{r,s})$ and $O_{\mathbb{Z}}(Q_{r,s})$ fixing a primitive vector u with $u^2 = 2M$. There is only one such vector up to equivalence, by the Nikulin embedding theorem [21, 24]. Therefore, the conjugacy class of the subgroup only depends on M and, by abuse of notation, we denote a particular subgroup by $O_{\mathbb{R}}(Q_{r,s}, M)$, $O_{\mathbb{Z}}(Q_{r,s}, M)$, respectively. Then we have

$$\mathcal{M}(S^MX) \cong O_{\mathbb{Z}}(Q_{r,s}, M)\backslash O_{\mathbb{R}}(Q_{r,s}, M)/O(r+8s) \times O(r-1) \tag{3.6}$$

where $(r, s) = (5, 0)$ for $X = T4$ and $(r, s) = (5, 2)$ for $X = K3$. Note that $O_{\mathbb{R}}(Q_{r,s}) \cong O(r + 8s, r - 1)$ and so the real dimension is

$$\dim\left(\mathcal{M}(S^MX)\right) = (r-1)(r+8s) \tag{3.7}$$

The space $\mathcal{M}(S^MX)$ has real dimension $\mathcal{D} = 20$ for $X = T4$ and $\mathcal{D} = 84$ for $X = K3$. The "extra" four dimensions relative to the moduli spaces for $T4$ and $K3$, respectively, can be thought of as arising from the hypermultiplet of blow-up modes of the locus of A_1 singularities along the big diagonal (where some pair of points coincides) in the symmetric product orbifold. Thus a neighborhood U in $\mathcal{M}(X)$ determines a corresponding neighborhood $\cong U \times \left(\mathbb{R}^3 \times S^1\right)$ in $\mathcal{M}(S^MX)$ for $M > 1$ where the factor of $\mathbb{R}^3$ represents levels of three hyperkähler moment maps and the S^1 is a period of a B-field. [4]

Let $v_X^{\rm tr}(M)$ denote the volume of the moduli space for S^MX in the homogeneous metric induced from $ds^2 = \mathrm{Tr}_D(\mathcal{E}^{-1}d\mathcal{E})^2$. According to equation (2.20) the Zamolodchikov metric is related by rescaling with the factor $1/(2\pi^2)$ and so volumes are rescaled by $1/(\pi\sqrt{2})^{\mathcal{D}}$ where $\mathcal{D}$ is the dimension of the moduli space. In addition, in a symmetric product orbifold with M factors the Zamolodchikov metric is also rescaled by a factor of M. We justify this last statement as follows:

In general there is an immersion (most likely, an embedding) of moduli spaces

$$\iota : \mathcal{M}(\mathcal{C}) \hookrightarrow \mathcal{M}(\mathrm{Sym}^M(\mathcal{C})). \tag{3.8}$$

Indeed, if a point in $\mathcal{M}(\mathcal{C})$ has action S then the action for the corresponding point in $\mathcal{M}(\mathrm{Sym}^M(\mathcal{C}))$ is just

$$S^{(1)} + \cdots + S^{(M)}, \tag{3.9}$$

[3]One proves this last statement using the Künneth formula and the fact that $h^{1,p}(X_i)$ vanishes unless $p = 1$ or $p = \dim_{\mathbb{C}} X_i - 1$. This in turn is proved from the Lefshetz hyperplane theorem. This argument certainly leaves room for a finite quotient in the relation between $\mathcal{M}(\mathcal{C}_1 \times \mathcal{C}_2)$ and $\mathcal{M}(\mathcal{C}_1) \times \mathcal{M}(\mathcal{C}_2)$. The presence of such finite quotients would considerably complicate the methods of Section §5, but we believe that it will not materially affect the main conclusions of that section. In any case, we leave this as an issue that deserves further thought.

[4]The case of S^1T4 deserves one further comment. In general, resolving the 16 fixed points of $T4/\mathbb{Z}_2$ results in 16×4 conformal field theory moduli for a total of $16 \times 5 = 80$ moduli. However, in $\mathcal{M}(S^1T4)$ we preserve translation invariance in the resolution, so there are only $16 + 4 = 20$ conformal field theory moduli.

where $S^{(i)}$ is the action S for the i^{th} factor. Note that there is no overall factor involving a power of M. (This last statement can be verified by noting that the stress-energy tensor of $\mathrm{Sym}^M(\mathcal{C})$ is the sum of the stress-energy tensors of $\mathcal{C}$. Recall then that the energy-momentum tensor is the variation of the action with respect to the worldsheet metric.) This immersion is simple enough, when viewed from the CFT perspective, but is rather more nontrivial when understood in terms of moduli spaces of hyperkähler metrics. In any case, we claim that, when restricted to the image of ι in (3.8) we have the commutative diagram:

$$\begin{tikzcd} V^{1,1}(\mathrm{Sym}^M(\mathcal{C})) \arrow[r, "\Psi"] & T\mathcal{M}(\mathrm{Sym}^M(\mathcal{C})) \\ V^{1,1}(\mathcal{C}) \arrow[u, "\varphi"] \arrow[r, "\Psi"] & T\mathcal{M}(\mathcal{C}) \arrow[u, "\iota_*"'] \end{tikzcd} \tag{3.10}$$

where

$$\varphi(\mathcal{O}) = \mathcal{O}^{(1)} + \cdots + \mathcal{O}^{(M)}. \tag{3.11}$$

Again, the absence of a normalization factor involving a power of M follows from the definition in terms of the deformation of the action, together with the additivity of the actions noted above. Therefore,

$$\iota^*(ds^2_{\mathrm{Z}}(\mathrm{Sym}^M(\mathcal{C}))) = M ds^2_{\mathrm{Z}}(\mathcal{C}). \tag{3.12}$$

In our application to $\mathcal{M}(S^M X)$, the metric is again a homogeneous metric, so the scale is determined by restricting to the subspace isomorphic to $\mathcal{M}(S^1 X)$. Letting $v_X^{\mathrm{Zam}}(M)$ denote the volume in the Zamolodchikov metric we have:

$$v_X^{\mathrm{Zam}}(M) = \begin{cases} \left(\frac{M}{2\pi^2}\right)^{42} v_X^{\mathrm{tr}}(M) & X = K3, \quad M > 1 \\ \left(\frac{M+1}{2\pi^2}\right)^{10} v_X^{\mathrm{tr}}(M) & X = T4, \quad M \geq 1 \end{cases}. \tag{3.13}$$

We will return to these three ensembles, and their Zamolodchikov measures in Section 5 below.

4. Some Results From Number Theory

4.1 Volumes For $\mathcal{N}_{r+8s,r}$

The volumes for the spaces $\mathcal{N}_{r+8s,r}$ are related to the so-called "mass" of a genus of lattices determined by $Q_{r,s}$. [5] The "mass" of a genus of lattices was introduced in the work of Carl Ludwig Siegel. In the case with $r > 0$ there is only one equivalence class in the genus and the mass can be identified as the volume of a fundamental domain:

$$Mass(L_{r,s}) = \int_{O_{\mathbb{Z}}(Q_{r,s})\backslash O_{\mathbb{R}}(Q_{r,s})} \mu \tag{4.1}$$

where μ is a left-right invariant measure. Siegel gave general formulae for this and related expressions. Siegel's formulae involve products of "local density factors" over all prime

[5] The word "mass" is a mistranslation of the German word "mass." The correct translation is "measure," a term that makes much better sense. But the term "mass" has become standard.

numbers, and the computation of those densities is itself nontrivial. In the papers [4, 14] the relevant factors were computed for the general case of unimodular lattices (not necessarily even). The paper [4] gives the volume using the measure $\mu = \mu^{\rm cpt}$ normalized so that its analytic continuation to the connected compact group $SO(D)$ gives unit total volume:

$$\mathrm{vol}^{\,\rm cpt}(SO(D)) = 1 \tag{4.2}$$

where $D = 2r + 8s = 2d$. [6] In particular, applying Theorem 3.1 of [4] to our case gives

$$Mass(L_{r,s}) = 2(d-1)!\frac{\zeta(d)}{(2\pi)^d}\prod_{j=1}^{d-1}\frac{|B_{2j}|}{4j}. \tag{4.3}$$

Readers who wish to check this specialization should note that the Tamagawa number $\tau(G) = 2$ since G is an orthogonal group, $d_F = 1$ and $\deg(F) = 1$ since $F = \mathbb{Q}$, and likewise $disc(q) = +1$. For the local factors λ_p for all the odd primes p we consult Table 3, p. 117 of [14]. We are in the case $\delta = 1$ and the Hasse-Minkowski-Witt invariant $w = 1$, because all the Hilbert symbols are $(\pm 1, \pm 1) = 1$. Therefore we have the first line of the table and $\lambda_p = 1$. For the prime $p = 2$ we consult Corollary 5.3 of [4] and again, $\lambda_{p=2} = 1$. Finally, we used $\zeta(2j)/(2\pi)^{2j} = |B_{2j}|/(2(2j)!)$ to rewrite the formula slightly.

Now we have to convert to the measure $\mu^{\rm tr}$. We do this by noting that $O(n+1)/O(n) \cong S^n$, the n-dimensional sphere. The sphere S^n of radius R has volume

$$R^n\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \tag{4.4}$$

in the standard round metric. A small computation shows that the homogeneous metric induced from $-\mathrm{Tr}_{n+1}(g^{-1}dg)^2$ gives the round metric with radius $\sqrt{2}$ and hence

$$\mathrm{vol}^{\,\rm tr}(O(D)) := \sigma(D) = 2^{(D+1)/2}\prod_{j=1}^{D-1}\left(\frac{(2\pi)^{\frac{j+1}{2}}}{\Gamma(\frac{j+1}{2})}\right) \tag{4.5}$$

where $\mathrm{vol}^{\,\rm tr}(O(D))$ is the volume in the metric $-\mathrm{Tr}_D(g^{-1}dg)^2$.

Combining the above remarks we have:

$$\mathrm{vol}^{\,\rm tr}(\mathcal{N}_{r+8s,r}) = \frac{\sigma(2r+8s)}{\sigma(r)\sigma(r+8s)}\cdot 2(d-1)!\frac{\zeta(d)}{(2\pi)^d}\prod_{j=1}^{d-1}\frac{|B_{2j}|}{4j} \tag{4.6}$$

In arriving at this formula there are two canceling factors of two. We have multiplied by the volume of $O(D)$ (giving an "extra" factor of two). However the element $(-1,-1) \in O(r) \times O(r+8s)$ does not act effectively on the quotient $O_{\mathbb{Z}}(Q_{r,s})\backslash O_{\mathbb{R}}(Q_{r,s})$, so when dividing by $\sigma(r)\sigma(r+8s)$ we have divided by an "extra" factor of two. [7]

[6] Reference [4] does not state whether the "compact form" is $O(D)$ or $SO(D)$. The computation for the special case $r = 2, s = 0$ in the appendix suggests that $SO(D)$ is what was meant. This interpretation has kindly been confirmed by M. Belolipetsky.

[7] In [1] Ashok and Douglas cited a related computation of Siegel for the volume of the moduli space of complex structures on a torus. The above formula for the volume of $\mathcal{N}_{r,r}$ differs from the formula they use since the Kähler modes and B-fields are not included in their formula.

4.2 Volumes For $S^M X$

Now we turn to the rather more challenging case of computing

$$\text{vol } (O_{\mathbb{Z}}(Q_{r,s}, M)\backslash O_{\mathbb{R}}(Q_{r,s}, M)) \tag{4.7}$$

where the notation was defined above equation (3.6). For applications to the ensembles of Section §3 we will specialize to $r = 5$ and $s = 2$ and $r = 5$ and $s = 0$.

The paper of C.L. Siegel [27] computes the volume of $O_{\mathbb{Z}}(Q_{r,s}, M)\backslash O_{\mathbb{R}}(Q_{r,s}, M)$ in a measure μ^{cls} normalized so that

$$\text{vol}^{\text{ cls}}(O(D)) = \prod_{j=1}^{D} \frac{\pi^{j/2}}{\Gamma(j/2)}. \tag{4.8}$$

Therefore we will have to take into account a fudge factor

$$\text{vol}^{\text{ tr}}(O(D)) = 2^{D(D+3)/4}\text{vol}^{\text{ cls}}(O(D)). \tag{4.9}$$

As long as $r > 1$, so that there is only one class of primitive vector u, the Siegel formula reduces to:

$$\frac{\text{vol}^{\text{ cls}}(O_{\mathbb{Z}}(Q_{r,s}, M)\backslash O_{\mathbb{R}}(Q_{r,s}, M))}{\text{vol}^{\text{ cls}}(O_{\mathbb{Z}}(Q_{r,s})\backslash O_{\mathbb{R}}(Q_{r,s}))} = \prod_{p<\infty} \alpha_p(M) \tag{4.10}$$

where the product is over all the finite primes,

$$\alpha_p(M) := \lim_{t\to\infty} \frac{A(d, M, p^t)}{p^{t(2d-1)}}, \tag{4.11}$$

and $A(d, M, p^t)$ is the number of representatives of $2M$ by $Q_{r,s}$ over the ring $\mathbb{Z}/p^t\mathbb{Z}$. That is:

$$A(d, M, p^t) = \#\{v \text{ mod} p^t | Q_{r,s}(v) = 2M \text{ mod} p^t\} \tag{4.12}$$

We now evaluate the "local densitites" $\alpha_p(M)$. To do this, we need a formula for $A(d, M, p^t)$.

The first remark is that the lattice $L_{r,s}$ is equivalent over the p-adic integers, for all finite primes p, to the simpler lattice $U^{\oplus d}$. For an odd prime p this follows, for example, from Theorem 3.1, p.115 of Cassels' book [6]. (Alternatively, one can apply Theorems 2 and 9 of Chapter 15 of [7].) The prime $p = 2$ is more delicate. Using the classification over 2-adic integers described in [21, 24] and unimodularity we conclude that the forms are also equivalent over the 2-adic integers. Therefore, $A(d, M, p^t)$ only depends on r, s through the combination $d = r + 4s$, as indicated in the notation. Moreover, we can replace (4.12) with the much simpler expression:

$$A(d, M, p^t) = \#\{x_i, y_i \text{mod} p^t | \sum_{i=1}^{d} 2x_i y_i = 2M \text{ mod} p^t\}. \tag{4.13}$$

We find this surprising: The only thing standing between the bland and boring $U^{\oplus d}$ and the arresting and attractive $L_{r,s}$ with its beautiful E_8 summands is the prime at infinity!

Next, we remark that, for each prime p the answer only depends on the power of p that divides M. So it suffices to compute $A(d,p^e,p^t)$, where $e \geq 0$, and we only consider $t > e$. We now have the result that for p an *odd* prime:

$$A(d,p^e,p^t) = p^{(2d-1)t}(1-p^{-d})\frac{1-p^{-(e+1)(d-1)}}{1-p^{-(d-1)}} \tag{4.14}$$

while for $p=2$ we have

$$A(d,2^e,2^t) = 2\cdot 2^{(2d-1)t}(1-2^{-d})\frac{1-2^{-(e+1)(d-1)}}{1-2^{-(d-1)}} \tag{4.15}$$

These formulae also apply for $d=1$ if we interpret the final quotient using L'Hopital's rule.

We now prove (4.14) and (4.15).

4.2.1 Proof Of The Formula For $A_p(d,p^e,t)$

To begin, we consider the case of an odd prime. Let us write (4.14) as

$$p^{(2d-1)(t-e-1)+de+(d-1)}(p^d-1)\big[1+p^{d-1}+(p^{d-1})^2+\cdots+(p^{d-1})^e\big] \tag{4.16}$$

This makes it clear that it is an integer in the range $t > e$ where it is meant to hold. We divide the set of solutions $\mathcal{S}$ into two disjoint sets $\mathcal{S} = \mathcal{S}_1 \amalg \mathcal{S}_2$ where $\mathcal{S}_1$ is the set of solutions where at least one of the x_i is invertible mod p^t, while $\mathcal{S}_2$ is the set of solutions where all of the x_i fail to be invertible mod p^t.

We first show that the number of solutions in $\mathcal{S}_1$ is:

$$p^{t(2d-1)}(1-p^{-d}). \tag{4.17}$$

To do this we further divide up $\mathcal{S}_1$ into a disjoint union of sets $\mathcal{S}_1^j$, with $j=1,\ldots,d$. The set $\mathcal{S}_1^j$ is the set of solutions such that x_j is invertible mod p^t, but x_i for $i<j$ are not invertible mod p^t. The number of invertible elements in $\mathbb{Z}/p^t\mathbb{Z}$ is p^t-p^{t-1} and the number of noninvertible ones is p^{t-1}. When x_j is invertible, we can solve for y_j. Therefore, the total number of solutions in $\mathcal{S}_1$ is just

$$(p^t-p^{t-1})(p^t)^{d-1}\sum_{j=1}^{d}(p^{t-1})^{(j-1)}(p^t)^{d-j} = p^{t(2d-1)}(1-p^{-d}) \tag{4.18}$$

where (p^t-p^{t-1}) is the number of invertible choices for x_j, $(p^{t-1})^{(j-1)}$ is the number noninvertible choices for x_i with $i<j$, $(p^t)^{d-j}$ is the number of arbitrary choices for x_i with $i>j$, and $(p^t)^{d-1}$ is the number of choices for y_k, with $k\neq j$. Note that for the case $e=0$ the set of solutions $\mathcal{S}_2$ is empty, since then $2M$ is prime to p, so we have proven (4.16) for $e=0$. [8]

[8] The argument can be also be used to count the number of solutions to $\sum_i x_i y_i = 0 \bmod p$ because the only other case is where all $x_i=0$. But then y_i can be anything, so we just add p^d, thus getting $p^{2d-1}-p^{d-1}+p^d$ solutions.

We can now prove the general result by induction on e. If $e > 0$ the set $\mathcal{S}_2$ will be nonempty. If all the x_i fail to be invertible then we can write $x_i = p\tilde{x}_i$ where $\tilde{x}_i$ is defined mod p^{t-1}. Moreover,

$$\sum_i \tilde{x}_i y_i = p^{e-1} \mathrm{mod} p^{t-1} \tag{4.19}$$

By the inductive hypothesis we know the number of solutions, modulo p^{t-1}, to this equation is

$$p^{(2d-1)(t-e-1)+d(e-1)+(d-1)}(p^d-1)\left[1+p^{d-1}+(p^{d-1})^2+\cdots+(p^{d-1})^{e-1}\right] \tag{4.20}$$

But now we need to lift the solutions $(\tilde{x}_i, y_i)\mathrm{mod}p^{t-1}$ to solutions mod p^t. The lifts of $\tilde{x}_i$ are $\tilde{x}_i + a_i p^{t-1}$ and do not change the value of $x_i = p\tilde{x}_i \mathrm{mod} p^t$. Moreover, *all* the lifts of $y_i \to y_i + b_i p^{t-1}$ solve the equation because $x_i(y_i + b_i p^{t-1}) = x_i y_i + \tilde{x}_i b_i p^t = x_i y_i \mathrm{mod} p^t$. So all p^d lifts of the vector $(y_1, \dots, y_d)$ produce solutions. So the number of solutions of type $\mathcal{S}_2$ is

$$p^{(2d-1)(t-e-1)+de+(d-1)}(p^d-1)\left[1+p^{d-1}+(p^{d-1})^2+\cdots+(p^{d-1})^{e-1}\right] \tag{4.21}$$

Combining this with the number of solutions of type 1 we arrive at the desired (4.16).

Now, turning to the case of $p = 2$, we note that if

$$\sum_{i=1}^{d} 2x_i y_i = 2 \cdot 2^e \mathrm{mod} 2^t \tag{4.22}$$

(where we recall that $M = 2^e u$, with u odd and $e \geq 0$) then

$$\sum_{i=1}^{d} x_i y_i = 2^e \mathrm{mod} 2^{t-1} \tag{4.23}$$

Moreover, given a solution of (4.23) with x_i, y_i defined modulo 2^{t-1} there are 2^{2d} distinct lifts of solutions to (4.22) modulo 2^t. On the other hand, the argument we gave for the odd primes works equally well for counting solutions of (4.23). (The only slightly subtle point is that the group of invertible elements of $\mathbb{Z}/2^t\mathbb{Z}$ is not cyclic for $t > 1$. Nevertheless, it is still of order $2^t - 2^{t-1} = 2^{t-1}$, and that is all we used.)

4.2.2 Answer For The Volumes

Given the formulae (4.14) and (4.15) we get the local densities:

$$\alpha_p(M) = \begin{cases} (1-p^{-d})\frac{1-p^{-(e_p(M)+1)(d-1)}}{1-p^{-(d-1)}} & p \neq 2 \\ 2 \cdot (1-p^{-d})\frac{1-p^{-(e_p(M)+1)(d-1)}}{1-p^{-(d-1)}} & p = 2 \end{cases} \tag{4.24}$$

where $e_p(M)$ is the p-adic valuation of M. That is: $M = \prod_p p^{e_p(M)}$.

The formula (4.10) becomes

$$\frac{\mathrm{vol}^{\,\mathrm{cls}}(O_{\mathbb{Z}}(Q_{r,s}, M)\backslash O_{\mathbb{R}}(Q_{r,s}, M))}{\mathrm{vol}^{\,\mathrm{cls}}(O_{\mathbb{Z}}(Q_{r,s})\backslash O_{\mathbb{R}}(Q_{r,s}))} = 2\zeta(d)^{-1} f_d(M) \tag{4.25}$$

where

$$f_d(M) = \prod_{p|M} \frac{1 - p^{-(e_p(M)+1)(d-1)}}{1 - p^{-(d-1)}} \tag{4.26}$$

Taking into account the fudge-factor going from $\mathrm{vol}^{\mathrm{cls}}$ to $\mathrm{vol}^{\mathrm{tr}}$ and defining

$$V_{r,s}(M) := \mathrm{vol}^{\mathrm{tr}}(O_{\mathbb{Z}}(Q_{r,s}, M)\backslash O_{\mathbb{R}}(Q_{r,s}, M)/(O(r+8s) \times O(r-1))) \tag{4.27}$$

we conclude that:

$$V_{r,s}(M) = \sqrt{8}\frac{\sigma(2r+8s)}{\sigma(r-1)\sigma(r+8s)} \frac{(d-1)!}{(4\pi)^d} \left(\prod_{j=1}^{d-1} \frac{|B_{2j}|}{4j} \right) f_d(M). \tag{4.28}$$

5. Application To Holography

5.1 General Strategy

In this subsection we review some of the considerations from Section VI of [5]. For more background see [5]. The considerations of [5] were motivated by the following question:

Consider a sequence $\mathcal{C}_M$ of conformal field theories (say, with $(4,4)$ supersymmetry and $c = 6M$). "How likely" is it that this sequence has a large M holographic dual with weakly coupled gravity?

Following the recent papers [3, 17, 19], reference [5] proposed that an important *necessary* criterion for the existence of such a holographic dual is that the elliptic genus should exhibit a Hawking-Page phase transition. Reference [5] further argued that a necessary criterion for a Hawking-Page phase transition is that the absolute value of the extremal polar coefficient, denoted $\mathfrak{e}(\mathcal{C}_M)$, must grow at most polynomially in M, that is, it is $o(\exp(M^\delta))$ for any $\delta > 0$.

In principle, we should define a probability measure on *sequences* of conformal field theories $\{\mathcal{C}_M\}$ drawn from the ensembles $\mathfrak{E}$ described in Section §3. We will not do that here. As a surrogate, we will instead state some (possibly M-dependent) property $\mathcal{P}$ of a CFT and instead consider sequences of probabilities p_M that CFT's of central charge M (drawn from the ensemble $\mathfrak{E}_M$) satisfy property $\mathcal{P}$. For example, $\mathcal{P}$ might be the statement that $\mathfrak{e}(\mathcal{C}) \leq \kappa M^\ell$. The measure in $\mathfrak{E}_M$ for $\mathcal{C}$ to have $\mathfrak{e}(\mathcal{C}) = \mathfrak{e}$ is

$$\mu(\mathfrak{e}; M) := \frac{\mathrm{vol}\,(\mathfrak{e}; M)}{\mathrm{vol}\,(M)} \tag{5.1}$$

where $\mathrm{vol}\,(M) = \mathrm{vol}\,(\mathfrak{E}_M)$ and $\mathrm{vol}\,(\mathfrak{e}; M)$ is the volume of the subset in $\mathfrak{E}_M$ of theories with $\mathfrak{e}(\mathcal{C}) = \mathfrak{e}$. Therefore, we introduce:

$$p_M(\kappa, \ell) = \sum_{\mathfrak{e} \leq \kappa M^\ell} \frac{\mathrm{vol}\,(\mathfrak{e}; M)}{\mathrm{vol}\,(M)}. \tag{5.2}$$

If the limit

$$\lim_{M \to \infty} p_M(\kappa, \ell) \tag{5.3}$$

exists and is independent of κ, we will say that it is the probability that a sequence of CFT's $\{\mathcal{C}_M\}$ drawn from the ensemble $\mathfrak{E}$ has $\mathfrak{e}$ growing at most like a power M^ℓ. We will denote it by $\mathfrak{p}(\ell)$.

If our ensemble is a multiplicative ensemble then it is useful to define *prime* CFT's to be those which are not a product (even up to deformation) of CFT's in $\mathfrak{E}$ with positive central charge. Denoting the prime CFT's at a fixed central charge $c = 6m$ by $\mathcal{C}_{m,\alpha}$ we can form the generating functional

$$\prod_{m=1}^{\infty} \prod_{\alpha} \frac{1}{1 - v_\alpha(m)\mathfrak{e}_\alpha(m)^{-s} q^m} = 1 + \sum_{M=1}^{\infty} \xi(s; M) q^M \tag{5.4}$$

where $\mathfrak{e}_\alpha(m)$ and $v_\alpha(m)$ are the extremal polar coefficient and volumes associated to $\mathcal{C}_{m,\alpha}$, respectively. The coefficient of q^M is

$$\xi(s; M) = \sum_{\mathfrak{e}=1}^{\infty} \frac{\mathrm{vol}\,(\mathfrak{e}; M)}{\mathfrak{e}^s} \tag{5.5}$$

and it gives the volumes $\mathrm{vol}\,(\mathfrak{e}; M)$. (For later use, note that $\xi(0; M) = \mathrm{vol}(M)$ is the total volume of the theories with fixed central charge $c = 6M$.)

5.2 Ingredients For The Three Ensembles

In the three ensembles we are considering the prime CFT's of index m are $S^m K3$, $S^m T4$, and $\{S^m K3, S^m T4\}$, respectively.

The extremal polar coefficient for $S^m K3$ is easily deduced from the formula for symmetric product orbifolds [8] and is

$$\mathfrak{e}(S^m K3) = m + 1. \tag{5.6}$$

For $S^m T4$ the relevant result can be deduced from [20] and is again

$$\mathfrak{e}(S^m T4) = m + 1. \tag{5.7}$$

(We provide a few details in Appendix B below.)

Let us denote the Zamolodchikov volumes of $\mathcal{M}(S^M X)$ by $v_X(M)$. Recalling equation (3.13) we have:

$$v_{K3}(M) = \begin{cases} \left(\frac{1}{2\pi^2}\right)^{40} \mathrm{vol}^{\,\mathrm{tr}}(\mathcal{N}_{20,4}) & M = 1 \\ \left(\frac{M}{2\pi^2}\right)^{42} V_{5,2}(M) & M > 1 \end{cases} \tag{5.8}$$

$$v_{T4}(M) = \left(\frac{M+1}{2\pi^2}\right)^{10} V_{5,0}(M) \qquad M \geq 1 \tag{5.9}$$

and now, thanks to equations (4.6) and (4.28), we have explicit results for these volumes. The numerical values are amusing. We have

$$v_{K3}(1) = \frac{(131)(283)(593)(617)(691)^2(3617)(43867)}{2^{40} \cdot 3^{34} \cdot 5^{15} \cdot 7^9 \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^3 \cdot 23 \cdot \pi^{40}} \cong 1.66 \times 10^{-61} \tag{5.10}$$

and for $M > 1$:

$$v_{K3}(M) = \rho M^{42} f_{13}(M) \tag{5.11}$$

with

$$f_{13}(M) = \prod_{p|M} \frac{1 - p^{-12-12e_p(M)}}{1 - p^{-12}} \tag{5.12}$$

where

$$\begin{aligned} \rho &= \frac{(103)(131)(283)(593)(617)(691)(3617)(43867)(2294797)}{2^{51} \cdot 3^{35} \cdot 5^{15} \cdot 7^{10} \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot \pi^{42}} \\ &\cong 5.815 \times 10^{-63} \end{aligned} \tag{5.13}$$

Similarly, we have

$$v_{T4}(M) = \rho'(M+1)^{10} f_5(M) \tag{5.14}$$

with

$$f_5(M) = \prod_{p|M} \frac{1 - p^{-4-4e_p(M)}}{1 - p^{-4}} \tag{5.15}$$

where

$$\begin{aligned} \rho' &= \frac{1}{2^{14} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot \pi^{10}} \\ &\cong 3.24 \times 10^{-18} \end{aligned} \tag{5.16}$$

5.3 Evaluating The Probabiity $\mathfrak{p}(\ell)$

Let us now return to the evaluation of $\mathfrak{p}(\ell)$ for the three ensembles. We consider the functions:

$$H_\ell(s) := \lim_{M\to\infty} (M+1)^{\ell s} \frac{\xi(s;M)}{\xi(0;M)} = \lim_{M\to\infty} \sum_{\mathfrak{e}=M+1}^{2^M} \frac{\mathrm{vol}\,(\mathfrak{e};M)}{\mathrm{vol}\,(M)} \left(\frac{(M+1)^\ell}{\mathfrak{e}}\right)^s \tag{5.17}$$

We claim that the limit exists and moverover, for all positive integers ℓ, it converges to the characteristic function:

$$\chi(s) = \begin{cases} 1 & s = 0 \\ 0 & s > 0 \end{cases}. \tag{5.18}$$

By splitting the sum in (5.17) into terms with $\mathfrak{e} \le \kappa(M+1)^\ell$ and $\mathfrak{e} > \kappa(M+1)^\ell$ it is easy to see that $(M+1)^{\ell s} \frac{\xi(s;M)}{\xi(0;M)} \ge \kappa^{-s} p_M(\kappa, \ell) \ge 0$, and hence if $H_\ell(s) = \chi(s)$ it must be that $\lim_{M\to\infty} p_M(\kappa, \ell) = 0$ for all κ and ℓ.

It is precisely in this sense that we mean that almost none of the sequences of CFT's drawn from the the three ensembles defined in Section §3 have weakly coupled holographic duals.

Our strategy for proving that $H_\ell(s) = \chi(s)$ is to note that we can organize the terms contributing to $\xi(s;M)$ in terms of partitions of M:

$$M = \lambda_1 + \cdots + \lambda_k \qquad\qquad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k. \tag{5.19}$$

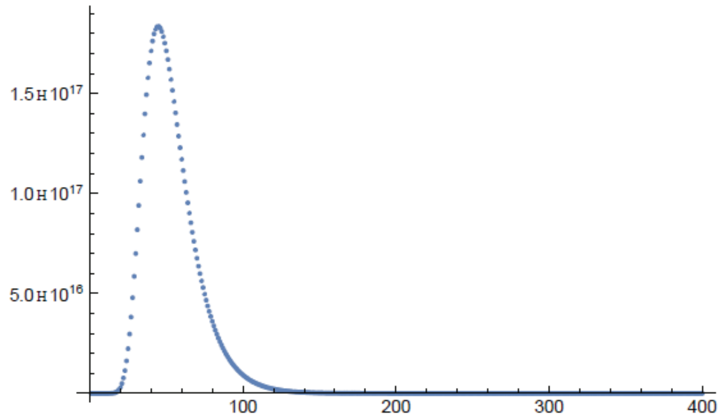

Figure 1: Showing the distribution of $p_k(n)$ as a function of k for $n = 400$. The Erdös-Lehner mean value of k, $\bar{k} = \frac{\sqrt{6}}{2\pi} 20 \log(20) \cong 46.7153$, is a very good approximation to the location of the maximum of the distribution. The actual maximum is at $k = 45$.

Expanding the N^{th} term in the product in (5.4) produces the parts in the partition with $\lambda_j = N$. We now ask: "What is the "typical" partition for large M?" This is an imprecise, and rather subtle question. To get some sense of an answer it is useful to consider the number $p_k(n)$ of partitions of n into precisely k parts (as in (5.19)). A generating function is [9]

$$\sum_{n=1}^{\infty} p_k(n) x^n = x^k \prod_{j=1}^{k} \frac{1}{1 - x^j} \tag{5.20}$$

and some naive estimates with Stirling's formula suggests that the distribution of $p_k(n)$ as a function of k for large n should be peaked around $k \cong \sqrt{n}$. This is, in part, confirmed by looking at numerical data. See, for example, Figure 1. One natural guess, then, is that the "typical" partition has $k \cong \sqrt{n}$ with "most of the parts" on the order of $\sqrt{n}$.

The above naive picture can be considerably improved using the statistical theory of partitions [13, 28, 29]. Without going into a lot of complicated asymptotic formulae, the main upshot is that $p_k(n)$ indeed is sharply peaked with a maximum around

$$\bar{k}(n) := \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n. \tag{5.21}$$

See Figure 1 for a numerical illustration. Moreover, according to [28, 29], and again speaking very roughly, the number of terms in the partition λ_j with $\lambda_j \cong \frac{\sqrt{6}}{2\pi}\sqrt{n}$ is order $\sqrt{6n}/\pi$.

Very roughly speaking, then, the dominant source of partitions of M for large M should have approximately $\bar{k}(M)$ parts with most of the parts of order $\sqrt{M}$. The key fact

[9]Note that the transpose of a partition with k parts is a partition whose largest part has size k. The generating function is more obvious from the latter viewpoint.

about $v_X(N)$ we need to know is that the dominant effect is the power N^{42} for $X = K3$ and N^{10} for $X = T4$. The arithmetic functions $f_{13}(N)$ and $f_5(N)$ do not change the value significantly. They are clearly bounded by N. [10] Therefore, $v_{K3}(N)$ has a growth larger than N^{42} and smaller than N^{43} and similarly for $X = T4$. The growth of these volumes with a large power of N might seem to pose a problem but the number of partitions with parts on the order of M is relatively exponentially small. We naively estimate $\xi(s;M)$ by taking $\bar{k}(M)$ parts of order $\lambda_j \sim c\sqrt{M}$ and thereby expect the asymptotics of $\xi(s;M)$ to be given by

$$\left(v_X(c\sqrt{M})\right)^{\bar{k}(M)} (c\sqrt{M})^{-s\bar{k}(M)} e^{2\pi\sqrt{\frac{M}{6}}} + \cdots \tag{5.22}$$

for some constant c. Therefore we expect the asymptotics of $\xi(s;M)/\xi(0;M)$ to be of the form

$$\frac{\xi(s;M)}{\xi(0;M)} \sim (c\sqrt{M})^{-s\bar{k}(M)} \tag{5.23}$$

Therefore, the limit defining $H_\ell(s)$ is given by

$$\lim_{M\to\infty} (M+1)^{\ell s} (c\sqrt{M})^{-s\bar{k}(M)} = \chi(s), \tag{5.24}$$

thus establishing our claim for the first two ensembles with $X = K3$ or $X = T4$.

For the third ensemble with X drawn from $\{K3, T4\}$ we need to look at pairs of partitions summing to M. When we enumerate such partitions the sum

$$\sum_{j=0}^{M} p(j)p(M-j), \tag{5.25}$$

where $p(n)$ is the orderinary partition function, has a saddle-point at $j = M/2$ so we expect the previous arguments to give, once again, $H_\ell(s) = \chi(s)$.

The above arguments are admittedly extremely rough. It would be worthwhile to prove the above claim more carefully.

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[10] A better bound is $f_d(N) \leq N^{C_d}$ with $C_d = \log[2^{d-1}/(2^{d-1}-1)]/\log 2$ giving $C_{13} = 0.000352263$ and $C_5 = 0.0931094$. To prove it note that $f_d(M) \leq \prod_{p|M}(1-p^{-(d-1)})^{-1}$, but $(1-p^{-(d-1)})^{-1} \leq p^{C_d}$ since $\frac{1}{x}\log[e^{(d-1)x}/(e^{(d-1)x}-1)]$ is a monotone decreasing function of x.

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A. An Explicit Computation Of The Volume Of $\mathcal{N}_{2,2}$

One can compute the volume of $\mathcal{N}_{2,2}$ explicitly using the fact that $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ double-covers the identity component of $O_{\mathbb{R}}(II^{2,2})$. (One must keep track of several tricky factors of two in this computation.) Using the generators of the duality group $O_{\mathbb{Z}}(II^{2,2})$ given in [16] one can check that the Narain space $\mathcal{N}_{2,2}$ is a quotient of a product of upper half-planes $\mathcal{F} \times \mathcal{F}$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$, acting via $(\tau,\rho) \to (\rho,\tau)$ and $(\tau,\rho) \to (-\bar{\tau},-\bar{\rho})$. On the other hand, the pullback of the metric $\mathrm{Tr}_{D=4}(\mathcal{E}^{-1}d\mathcal{E})^2$ is $2(\mathrm{Tr}(A^{-1}dA)^2 + \mathrm{Tr}(B^{-1}dB)^2)$ where $\mathcal{E}$ pulls back to a pair $(A,B) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Therefore,

$$\mathrm{vol}\,(O_{\mathbb{Z}}(Q)\backslash O_{\mathbb{R}}(Q)/O(2)\times O(2)) = (\sqrt{2})^4 \frac{1}{4} \mathrm{vol}\,(\mathcal{F}\times\mathcal{F}) = \mathrm{vol}\,(\mathcal{F}\times\mathcal{F}) \tag{A.1}$$

where the volume form on $\mathcal{F}$ is induced from the metric $\mathrm{Tr}(A^{-1}dA)^2$ on $SL(2,\mathbb{R})$. Using a standard KAN decomposition so that $\tau = x + \mathrm{i}y = A \cdot \mathrm{i}$ we find that this metric is half of the standard Poincaré metric, and $\mathrm{vol}\,(\mathcal{F}) = \pi/3$ in the standard metric. Putting all these facts together we get

$$\mathrm{vol}\,(O_{\mathbb{Z}}(Q_{2,2})\backslash O_{\mathbb{R}}(Q_{2,2})) = \frac{4\pi^4}{9} \tag{A.2}$$

in agreement with (4.3), provided we take (4.2).

B. Extremal Polar Coefficient For S^mT4

We use equation (5.16) of [20] to give the formula for the generating function of the elliptic genera of S^mT4:

$$\left(\sum_{m=0}^{\infty} p^{m+1}\mathrm{Ell}(q,y;S^mT4)\right) = \frac{\mathfrak{P}}{y-2+y^{-1}} \sum_{m\geq 1, n\geq 0, \ell\in\mathbb{Z}} \frac{\hat{c}(nm,\ell)p^m q^n y^\ell}{(1-p^m q^n y^\ell)^2} \tag{B.1}$$

where

$$\mathfrak{P} = \prod_{n=1}^{\infty} \frac{(1-q^n)^4}{(1-yq^n)^2(1-y^{-1}q^n)^4} \tag{B.2}$$

and

$$\sum_{n\geq 0,\ell\in\mathbb{Z}} \hat{c}(n,\ell)q^n y^\ell = -\frac{(\vartheta_1(z|\tau))^2}{\eta^6} = (y-2+y^{-1})\mathfrak{P}^{-1} \tag{B.3}$$

For the extremal polar coefficient we are interested in the coefficient of $p^m q^0 y^m$. Since we want the q^0 term we can set $q=0$ and then the right hand side of (B.1) reduces to

$$\sum_{m\geq 1} p^m \sum_{s|m} s \frac{y^s - 2 + y^{-s}}{y-2+y^{-1}} \tag{B.4}$$

so the q^0 term of $\mathrm{Ell}(q,y;S^mT4)$ is

$$\sum_{s|(m+1)} s\frac{y^s-2+y^{-s}}{y-2+y^{-1}} = (m+1)y^m+\cdots+(m+1)y^{-m} \tag{B.5}$$

and hence $\mathfrak{c}(S^mT4)=m+1$.

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